

THE TAYLOR SERIES OF THE GAUSSIAN KERNEL

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"From some people one can learn more than mathematics"

ABSTRACT. We describe a formula for the Taylor series expansion of the Gaussian kernel around the origin of $\mathbb{R}^n \times \mathbb{R}$.

1. INTRODUCTION

The explicit formulae for the power series expansion at the origin of the fundamental solution of the Laplace operator in \mathbb{R}^n , $n \geq 2$, are well known. In particular, when

$$\Gamma(x, y) = \frac{1}{\omega_n(2-n)} |x - y|^{2-n},$$

ω_n is the surface measure of the unit sphere \mathbb{S}^{n-1} in \mathbb{R}^n and $n > 2$, or when

$$\Gamma(x, y) = \frac{1}{2\pi} \log |x - y|$$

and $n = 2$, the following hold:

$$(1.1) \quad \varphi(x) = \int_{\mathbb{R}^n} \Gamma(x, y) \Delta \varphi(y) dy, \text{ when } \varphi \in C_0^\infty(\mathbb{R}^n),$$

$$(1.2) \quad \Gamma(x, y) = \sum_{k=0}^{+\infty} \frac{|x|^k}{|y|^{k+n-2}} Z_k(x' \cdot y'), \text{ when } 0 \leq |x| < |y|,$$

$$(1.3) \quad Z_k(x' \cdot y') = \begin{cases} -\frac{1}{2k+n-2} Z_{x'}^{(k)}(y') & , \text{ when } 2k+n-2 > 0, \\ \frac{1}{2\pi} \log |y| & , \text{ when } 2k+n-2 = 0. \end{cases}$$

Here, $x' = x/|x|$ and $Z_{x'}^{(k)}(y)$ is the *zonal harmonic of degree k* , i.e., the kernel of the projection operator of $L^2(S^{n-1})$ onto the spherical harmonics of degree $k \geq 0$. We recall that the spherical harmonics of degree $k \geq 0$ are the eigenfunctions of the spherical Laplacian on \mathbb{S}^{n-1} and corresponding to the eigenvalue, $k(k+n-2)$. See [8, (2.17)] for (1.1) and [11, Chapter IV] for (1.2) and (1.3).¹

For fixed z' in S^{n-1} and $k \geq 0$, the function

$$E_k(x) = |x|^k Z_k(x' \cdot z')$$

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¹A simple way to derive (1.2) and (1.3) is to solve, $\Delta u = f$ in \mathbb{R}^n , via the method of separation of variables in spherical coordinates and then, to compare the solution, which the latter method yields, with the one obtained via the convolution with the fundamental solution.

is a homogeneous harmonic polynomial of degree k , i.e.,

$$\Delta E_k = 0 \text{ and } E_k(\lambda x) = \lambda^k E_k(x) ,$$

when $\lambda \geq 0$ and x is in \mathbb{R}^n . The function,

$$|y|^{2-n-k} Z_k(z' \cdot y')$$

is harmonic in $\mathbb{R}^n \setminus \{0\}$ and is homogeneous of degree, $2 - k - n$. In fact, it is the Kelvin transformation of E_k . Here recall, that the Kelvin transformation v of a function u is

$$v(x) = |x|^{2-n} u(x/|x|^2) \quad , \quad \Delta v(x) = |x|^{-n-2} \Delta u(x/|x|^2) ,$$

$$\varphi = \sum_{k=0}^{+\infty} Z_k(\varphi) \quad , \quad \|\varphi\|_{L^2(\mathbb{S}^{n-1})}^2 = \sum_{k=0}^{+\infty} \|Z_k(\varphi)\|_{L^2(\mathbb{S}^{n-1})}^2 ,$$

when φ is in $C^\infty(\mathbb{S}^{n-1})$ and where

$$Z_k(\varphi) = \int_{\mathbb{S}^{n-1}} Z_{x'}^{(k)}(y') \varphi(y') dy' \text{ and } k \geq 0 .$$

Moreover, the formula in (1.2) is the Taylor series expansion of $\Gamma(x, y)$ around $x = 0$, for each fixed $y \neq 0$ in \mathbb{R}^n . (See [11, Chapter IV]).

The Taylor series (1.2) contains relevant information, which has had important applications in the mathematics of the last century and among others it has shown to be useful to obtain estimates leading to sharp results of strong and weak unique continuation for elliptic operators on \mathbb{R}^n . This can be seen in [9], [14] and [10].

The Gaussian kernel

$$G(x, t, y, s) = \begin{cases} (4\pi(t-s))^{-n/2} e^{-|x-y|^2/4(t-s)} & , \text{ when } s < t , \\ 0 & , \text{ when } s > t , \end{cases}$$

is the fundamental solution of the heat operator in \mathbb{R}^{n+1} , i.e.

$$f(x, t) = - \int_{-\infty}^t \int_{\mathbb{R}^n} G(x, t, y, s) (\Delta f - \partial_s f) dy ds , \text{ when } f \in C_0^\infty(\mathbb{R}^{n+1}) .$$

As far as the author knows (and this is rather surprising), it seems that nobody has written down and publish an explicit formula for the Taylor series expansion of $G(x, t, y, s)$ around the origin of \mathbb{R}^{n+1} , when (y, s) in \mathbb{R}^{n+1} , $s < 0$, is fixed. The purpose of this note is to fill in this gap.

To simplify the notation, we choose to give the formula for the Taylor series expansion of the fundamental solution of the backward heat equation,

$$(1.4) \quad G_b(x, t, y, s) = \begin{cases} (4\pi(s-t))^{-n/2} e^{-|x-y|^2/4(s-t)} & , \text{ when } t < s , \\ 0 & , \text{ when } t > s , \end{cases}$$

when s is positive and (y, s) in \mathbb{R}^{n+1} is fixed. The Taylor series for the Gaussian kernel follows from the identity

$$G(x, t, y, s) = G_b(x, -t, y, -s) .$$

The Hermite functions, h_k , are defined as

$$h_k(x) = (2^k k! \sqrt{\pi})^{-\frac{1}{2}} (-1)^k e^{x^2/2} \frac{d^k}{dx^k} (e^{-x^2}) , \quad k \geq 0 , \quad x \in \mathbb{R}$$

and $h_k = H_k(x) e^{-x^2/2}$, where H_k is a Hermite polynomial of degree k .

The Hermite functions on \mathbb{R}^n , ϕ_α , $\alpha = (\alpha_1, \dots, \alpha_n)$ in \mathbb{N}^n , are the product of the one-dimensional Hermite functions h_{α_j} , $j = 1, \dots, n$

$$\phi_\alpha(x) = \prod_{j=1}^n h_{\alpha_j}(x_j) .$$

They form a complete orthonormal system in $L^2(\mathbb{R}^n)$ and if, $H = \Delta - |x|^2$, is the Hermite operator, $H\phi_\alpha = -(2|\alpha| + n)\phi_\alpha$, where $|\alpha| = \alpha_1 + \dots + \alpha_n$. The kernel

$$\Phi_k(x, y) = \sum_{|\alpha|=k} \phi_\alpha(x)\phi_\alpha(y)$$

is the kernel of the projection operator of $L^2(\mathbb{R}^n)$ onto the Hermite functions of degree, $k \geq 0$, and when $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\varphi = \sum_{k=0}^{+\infty} P_k(\varphi) \quad , \quad \|\varphi\|_{L^2(\mathbb{R}^n)}^2 = \sum_{k=0}^{+\infty} \|P_k(\varphi)\|_{L^2(\mathbb{R}^n)}^2$$

and where

$$P_k(\varphi) = \int_{\mathbb{R}^n} \Phi_k(x, y)\varphi(y) dy \quad , \quad k \geq 0 .$$

The reader can find the proofs of the latter results in [13, Chapter 1].

What seems to be the counterpart of the Kelvin transformation in the parabolic setting is the Appell transformation v of a function u ([1], [2, pp. 282]):

$$v(x, t) = |t|^{-n/2} e^{-|x|^2/4t} u(x/t, 1/t)$$

and

$$\Delta v - \partial_t v = |t|^{-2-n/2} e^{-|x|^2/4t} (\Delta u + \partial_t u)(x/t, 1/t) .$$

The Appell transformation maps backward caloric functions into forward caloric functions.

A calculation shows that

$$Q_\alpha(x, t) = t^{k/2} \phi_\alpha(x/2\sqrt{t}) e^{|x|^2/8t} \quad , \quad \alpha \in \mathbb{N}^n \quad , \quad |\alpha| = k \quad ,$$

is backward caloric and in fact, it is a backward caloric polynomial in the (x, t) -variables, which is homogeneous of degree $k = |\alpha|$ in the parabolic sense, i.e.,

$$\Delta Q_\alpha + \partial_t Q_\alpha = 0 \text{ in } \mathbb{R}^{n+1} \text{ and } Q_\alpha(\lambda x, \lambda^2 t) = \lambda^k Q_\alpha(x, t) \quad ,$$

when $\lambda \geq 0$ and (x, t) is in \mathbb{R}^{n+1} (The later follows because a Hermite polynomial H_k is an even function, when k is even and an odd function, when k is odd). At the same time and in analogy with the what happens with the Kelvin transformation of the harmonic function, $|x|^k Z_k(z' \cdot x')$, the function

$$s^{-(k+n)/2} \phi_\alpha(y/2\sqrt{s}) e^{-|y|^2/8s} \quad ,$$

is forward caloric and is the Appell transformation of Q_α .

Having gathered all this data, it is possible to describe and write down the Taylor series expansion of the backward Gaussian kernel (1.4) at the origin of \mathbb{R}^{n+1} . We do it in the following theorem:

Theorem 1. *The following identity holds, when $t < s$, $s > 0$, and x, y are in \mathbb{R}^n*

$$(1.5) \quad G_b(x, t, y, s) = (4s)^{-n/2} e^{|x|^2/8t} \left(\sum_{k=0}^{+\infty} (t/s)^{k/2} \Phi_k(x/2\sqrt{t}, y/2\sqrt{s}) \right) e^{-|y|^2/8s} .$$

The proof of Theorem 1 is given in section 2 and it follows from a well known identity: the *generating formula* for the kernels, Φ_k (See (2.1) below).

The commentaries in [12, pp. 582–583], which are made with the purpose to explain the reader a simple approach to prove the identity (2.1) and in particular, the reference [6, pp. 335–336], show that *Theorem 1 was probably already known to some authors, though not explicitly written down and published*. In fact, the approach suggested in [12, pp. 582–583] to prove the identity (2.1), follows precisely the inverse path of the one we follow in section 2 to prove Theorem 1. Thus, Theorem 1 was probably known by W. Feller and E.M. Stein.

In the same way as the formula for the Taylor series of the fundamental solution of the Laplace operator has been useful to derive results of unique continuation for elliptic operators, the formula in Theorem 1 is what, in a certain sense, is behind the positive results of unique continuation for parabolic equations in [3], [5], [7] and [4].

The argument in section 2 gives a clue of how to proceed to find the Taylor series expansion of the fundamental solution of the Schrödinger operator, $\Delta + i\partial_t$, around the origin of \mathbb{R}^{n+1} and the corresponding building pieces of the solutions of the Schrödinger equation: “the *Schrödinger* homogeneous polynomials of degree $k \geq 0$ ”.

2. PROOF OF THEOREM 1

Proof. Recall the *generating formula* for the projection kernels, Φ_k [13]

$$(2.1) \quad \sum_{k=0}^{+\infty} \Phi_k(x, y) \xi^k = \pi^{-\frac{n}{2}} (1 - \xi^2)^{-\frac{n}{2}} e^{-\frac{1}{2} \frac{1+\xi^2}{1-\xi^2} (|x|^2 + |y|^2) + \frac{2\xi xy}{1-\xi^2}}, \text{ when } |\xi| < 1, \xi \in \mathbb{C}.$$

Replace x by $x/2\sqrt{t}$, y by $y/2\sqrt{s}$ and take $\xi = \sqrt{t/s}$ in (2.1), when $0 \leq t < s$. It gives

$$(2.2) \quad s^{-n/2} \sum_{k=0}^{+\infty} (t/s)^{k/2} \Phi_k(x/2\sqrt{t}, y/2\sqrt{s}) = \pi^{-\frac{n}{2}} (s-t)^{-\frac{n}{2}} e^{-\frac{s+t}{s-t} (|x|^2/8t + |y|^2/8s) + \frac{xy}{2(s-t)}}.$$

Then, multiply (2.2) by $4^{-n/2} e^{|x|^2/8t - |y|^2/8s}$ to get that the identity

$$(4s)^{-n/2} e^{|x|^2/8t} \left(\sum_{k=0}^{+\infty} (t/s)^{k/2} \Phi_k(x/2\sqrt{t}, y/2\sqrt{s}) \right) e^{-|y|^2/8s} = (4\pi(s-t))^{-\frac{n}{2}} e^{-|x-y|^2/4(s-t)}$$

holds, when $0 \leq t < s$, $s > 0$ and x, y are in \mathbb{R}^n , and Theorem 1 follows. \square

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